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The Simplicity of the Zeros of Norm-Minimizing Polynomials

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It is shown that polynomials which minimize certain L_s- or maximum norms defined on the real interval $I = [a, b]$ have only simple roots. This is also shown to be true for monotonically increasing functions of the above norms.

The following considerations refer to L_s- and maximum norms defined on the real interval $I = [a, b]$. The L_s-norms are of the form

$$\|f\|_s := \left\{\int_I |f(x)|^s \, du(x)\right\}^{1/s}, \qquad s \geqslant 1, \tag{1}$$

where $u(x)$ is for $s > 1$ nondecreasing and continuous on I with $u(b) > u(a)$, and for $s = 1$ continuously differentiable with positive derivative. The maximum norms are of the form

$$\|f\|_\infty := \max_{x \in I} \{w(x) \, |f(x)|\} \tag{2}$$

with $w(x)$ continuous and positive on I.

The problem of finding a polynomial which belongs to the set

$$D_n := \{p: p(x) = x^n + a_1 x^{n-1} + \cdots + a_n \,,\ a_i \in \mathbb{R} \text{ or } \mathbb{C},\ i = 1(1)n\} \tag{3}$$

and minimises the norms (1) or (2) over this set has a unique solution. For $s > 1$ this is due to the strict convexity of the norms (1). For $s = 1$ and for the maximum norms this is due to the fact that the powers of x multiplied by a positive continuous function make up a Haar system [1, pp. 81, 219].

In the case $s = 2$, i.e., the case of a Hilbert-norm, it is easy to show by means of Bessel's inequality that the solution of the problem is the polynomial of D_n which belongs to the set of orthogonal polynomials with respect to the corresponding scalar product [2, p. 39]. Polynomials, which minimize other L_s-norms, can therefore be considered as a generalization of the orthogonal polynomials [2, p. 41], and it may be expected that they have some properties in common with the orthogonal polynomials.

An important property of this kind is that the zeros of norm-minimizing polynomials lie all in I. This follows immediately from a more general theorem of Fejèr [3, pp. 243–244], and is simply due to the fact that if $p(x) = (x - x_1)\, q(x)$ is a polynomial with a zero x_1 not lying in I, then the polynomial $r(x) = (x - x_0)\, q(x)$, where x_0 is the nearest to x_1 point of I, has a smaller L_s- or maximum-norm since

$$| x - x_0 | < | x - x_1 | \qquad \forall x \in I. \tag{4}$$

Here it will be shown that the polynomials, which minimize the norms (1) and (2), have the further common property that all their zeros are simple. However, it should be noted that no counterpart of this property exists in the case that a complex curve C is considered instead of a real interval. For example, if C is the unit circle, then the minimizing polynomials of the corresponding L_s-norms with $u(x) = x$, and of the maximum-norm with $w(x) = 1$ are simply $p(x) = x^n$ [4, pp. 238, 240].

The theorem to be proved is the following:

THEOREM 1. *Let $p \in D_n$ be a polynomial which minimizes one of the norms* (1) *or* (2). *Then all the zeros of p are simple.*

Proof. Let the minimizing polynomial for $\| \ \|_s$ $(s \geqslant 1)$ have a multiple zero $c \in (a, b)$. Since this zero is at least double,

$$p(x) = (x - c)^2\, q(x), \qquad q \in D_{n-2}. \tag{5}$$

Furthermore, the following inequalities are valid:

$$(x - c)^2 > (x - (c - h))(x - (c + h)) = (x - c)^2 - h^2 := y(x) \tag{6}$$

with arbitrary positive h, and

$$y(x) \geqslant 0 \qquad \text{for} \quad x \in G := [a, c - h] \cup [c + h, b], \tag{7}$$

as well as

$$| y(x)| \leqslant h^2 \qquad \text{for} \quad x \in [c - h, c + h]. \tag{8}$$

For $s \geqslant 1$, and $y(x) \geqslant 0$, i.e., $x \in G$,

$$| x - c |^{2s} = (y(x) + h^2)^s \geqslant y(x)^s + h^{2s}. \tag{9}$$

Therefore, since $u(x)$ is nondecreasing, it follows that

$$\int_I | p(x)|^s \, du(x) \geqslant \int_G (y(x) + h^2)^s \, | q(x)|^s \, du(x)$$

$$\geqslant \int_G | y(x)\, q(x)|^s \, du(x) + h^{2s} \int_G | q(x)|^s \, du(x), \tag{10}$$

and hence

$$\int_I |p(x)|^s \, du(x) \geqslant \int_I |y(x)\, q(x)|^s \, du(x) - \int_{c-h}^{c+h} |y(x)\, q(x)|^s \, du(x) + h^{2s} \int_G |q(x)|^s \, du(x). \tag{11}$$

But for h small enough,

$$\int_{c-h}^{c+h} |y(x)\, q(x)|^s \, du(x) \leqslant h^{2s} \int_{c-h}^{c+h} |q(x)|^s \, du(x) < h^{2s} \int_G |q(x)|^s \, du(x). \tag{12}$$

The first of the inequalities (12) is true because of (8). The second is true for h small enough, because with h becomming smaller the integral over $[c - h, c + h]$ tends to zero, and the integral over G tends to the integral over I, which is positive.

From (11) and (12) it then follows that

$$\int_I |p(x)|^s \, du(x) > \int_I |y(x)\, q(x)|^s \, du(x), \tag{13}$$

that is,

$$\| p \|_s > \| yq \|_s , \tag{14}$$

in contradiction to the assumption that p minimizes the norm.

In the case $c = a$ or $c = b$ the proof is similar. The only difference is that instead of the interval $[c - h, c + h]$, the intervals $[a, a + h]$ resp. $[b - h, b]$ have to be considered.

In the case of a maximum-norm one has for sufficiently small h,

$$\begin{aligned} \max_{x\in I} \{(x - c)^2 \,|\, q(x)|\, w(x)\} &= \max_{x\in G} \{(x - c)^2 \,|\, q(x)|\, w(x)\} \\ &> \max_{x\in G} \{ y(x) \,|\, q(x)|\, w(x)\} \\ &= \max_{x\in I} \{|\, y(x)| \,|\, q(x)|\, w(x)\}. \end{aligned} \tag{15}$$

The maxima over I cannot lie in $[c - h, c + h]$ for h small enough because both expressions considered then become arbitrarily small in this interval.

From (15) it follows that

$$\| p \|_\infty > \| yq \|_\infty , \tag{16}$$

that is, an inequality which contradicts the assumption that any polynomial of the form (5) would minimize the maximum-norm.

It should be noted that if uniqueness of the minimizing polynomial is not required, then the assumptions for the L_1-norms can be the same as for the other L_s-norms. That is, $u(x)$ need only be continuous and nondecreasing with $u(b) > u(a)$. The uniqueness of the minimizing polynomial is then not guaranteed, but it is easy to see that all minimizing polynomials must have real and simple zeros lying in I. The L_1-norms defined thus can always be reduced, as shown above, by replacing any complex zero by a real zero lying in I, and any multiple real zero by simple ones.

The proof of Theorem 1 is independent of the specific value of s or of the specific $u(x)$ or $w(x)$ used, provided that the general assumptions are fulfilled. This means that, if p is a polynomial with a zero not lying in I, then all possible norms of the forms (1) and (2) can simultaneously be reduced by replacing this zero by the nearest point to it of I. Also if p has a multiple real zero in I, then, according to the inequalities (14) and (16), all these norms can simultaneously be reduced by replacing a quadratic factor of the polynomial by two simple factors. Therefore, the following general statement is valid:

THEOREM 2. *Let $M(t_1, \dots, t_m)$ be a monotonically increasing function of the nonnegative variables t_i, $i = 1(1)m$, and consider the functional*

$$G(f) := M(\|f\|_{s_1}, \dots, \|f\|_{s_m}), \tag{17}$$

where $\|f\|_{s_i}$ are norms of the forms (1) *and* (2), *possibly with different $u(x)$ or $w(x)$ for each norm. Then, if there exist polynomials $p \in D_n$ minimizing this functional over D_n, the zeros of these polynomials are all real and simple and lie in I.*

Proof. As stated above, any polynomial $p \in D_n$ with zeros not lying in I or with multiple zeros in I can be replaced by a polynomial $r \in D_n$ having only simple zeros all lying in I, whose norms are smaller ($\|r\|_s < \|p\|_s$ for all $s \geqslant 1$ and $s = \infty$). Since M is a monotonically increasing function, the value of the functional for this polynomial is also smaller ($G(r) < G(p)$). Therefore the minimizing polynomials of G have only real and simple zeros lying in I.

An example of a function M for which the functional $G(f)$ is again a norm is (with $t_i \geqslant 0$)

$$M(t_1, \dots, t_m) = (t_1^d + t_2^d + \cdots + t_m^d)^{1/d}, \qquad d \geqslant 1. \tag{18}$$

If one of the norms $\|f\|_{s_i}$ used is strictly convex, then the resulting norm $\|f\| := G(f)$ is also strictly convex, and has therefore a unique minimizing polynomial whose zeros are, according to the above theorem, simple and lie in I.

The strict convexity of the norm follows directly from the observation that, since M is a monotonically increasing function, the equality

$$\|f + g\| = \|f\| + \|g\| \tag{19}$$

is only possible if such an equality is valid for each of the norms $\| \ \|_{s_i}$, i.e.,

$$\|f + g\|_{s_i} = \|f\|_{s_i} + \|g\|_{s_i}, \qquad i = 1(1)m. \tag{20}$$

Since one of these norms is strictly convex, it follows that

$$g = cf, \qquad c > 0, \tag{21}$$

with c constant, which means that the above composite norm is also strictly convex.

The above example shows that Theorem 2 is a partial completion of Fejèr's theorem given by Davis [3, p. 244]. From Fejèr's theorem it follows that in this case the zeros of the minimizing polynomial are real and lie in I; from Theorem 2 it follows that they are also simple.

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